Hybrid h- and p-Type Multiplicative Schwarz (h-p-MUS) Preconditioned Algorithm of Higher-Order FE-BI-MLFMA for 3D Scattering

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***Abstract*—The key to design algorithms of higher-order finite element-boundary integral-multilevel fast multipole algorithm (FE-BI-MLFMA) is how to construct preconditioners for the FEM matrix part in the FE-BI matrix equation. A hybrid h- and p-Type multiplicative Schwarz (h-p-MUS) preconditioner is proposed for 3D scattering in this paper. First, the hierarchical higher-order FEM matrix is rewritten by classifying unknowns into different groups corresponding to different FEM orders. Then a preconditioner is constructed by the p-type multiplicative Schwarz method (p-MUS). To further improve efficiency in the p-MUS construction, an h-type multiplicative Schwarz preconditioner is employed for different submatrices. Numerical experiments show that this preconditioning scheme can offer a better efficiency both in CPU time and memory requirement than previous algorithms. This h-p-MUS preconditioner for the FEM matrices is applied to different algorithms of FE-BI-MLMFA. Analysis and Numerical results show that the h-p-MUS preconditioned conventional algorithm (h-p-MUS-CA) has a better efficiency over other h-p-MUS preconditioned algorithms. A variety of numerical experiments are performed for complex objects in this paper, demonstrating that this h-p-MUS-CA exhibits superior efficiency in the CPU time and memory, and greatly improves the capability of the higher-order FE-BI-MLFMA.**

I. INTRODUCTION

IT has been shown that the hybrid finite element-boundary integral-multilevel fast multipole algorithm (FE-BI-MLFMA) is a general, accurate, and efficient computing technique for open region problems such as scattering/radiation problems. Since the FEM seriously suffers from the numerical dispersion error, especially for large scale computational domain, hence the higher-order FEM has to be employed for successfully modeling large problems [1]-[3].

How to efficiently handle the FEM part of the FE-BI matrix is the key to design algorithms of FE-BI-MLFMA. A p-type multiplicative Schwarz (p-MUS) preconditioner was presented for the hierarchical higher-order FEM matrix in [4]. In the p-MUS preconditioner, the FEM unknowns are classified into different groups corresponding to different FEM orders. Then a preconditioner is constructed by the p-type multiplicative Schwarz method (p-MUS). In this construction, conventional ILU preconditioning techniques are usually performed to obtain the approximate inverses of submatrices. The computational efficiency of conventional ILU preconditioners still often cannot be promised for the FEM matrices generated form vector wave equations, especially when the solution domain becomes electrically large. Recently, the algebraic multilevel inverse-based ILU preconditioner (MIB-ILU) was proposed for solving indefinite sparse FEM linear systems [5]. By employing the graph partition technique, this method reorders the original FEM matrix into the hierarchical multilevel structure. Next, the inverse-based ILU dropping strategy is adopted to construct a robust preconditioner. It has been verified that the MIB-ILU exhibits superior efficiency in the CPU time and memory. This method actually is an h-type multilevel multiplicative Schwarz (h-MUS) preconditioning approach.

A hybrid h- and p-type MUS preconditioning algorithm (h-p-MUS) is proposed for hierarchical higher-order FEM matrices in this paper. In this h-p-MUS algorithm, the FEM unknowns are classified into different groups corresponding to different FEM orders. Then the p-type multiplicative Schwarz type preconditioner is applied to this rewritten FEM matrix. For computing the inverses of submatrices required in the p-MUS preconditioner construction, the h-MUS preconditioning approach is performed to obtain approximate inverses instead of the conventional ILU. Numerical experiments show that this h-p-MUS preconditioner has superior efficiency in both memory requirement and CPU time to previous preconditioners.

II. H-P-MUS PRECONDITIONED ALGORITHMS OF FE-BI-MLFMA

It is known that the problem of scattering by a complex target can be discretized by the hybrid FE-BI method [1] as.

$$\begin{bmatrix} K_{II} & K_{IS} & 0 \\ K_{SI} & K_{SS} & B \\ 0 & P & Q \end{bmatrix} \begin{Bmatrix} E_I \\ E_S \\ H_S \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ b \end{Bmatrix} \tag{1}$$

where $[K_{II}]$, $[K_{SS}]$, $[K_{IS}]$, $[K_{SI}]$, $[B]$ are sparse FEM matrixes, $[P]$ and $[Q]$ are dense BI matrices. Since the FEM seriously suffers from the numerical dispersion error, especially for large scale computational domain, hence the higher-order FEM has to be employed for successfully modeling large problems.

For large number of unknowns, equation (1) has to be solved by iterative solvers since the computational complexity of the matrix-multiplication in iterative solving procedure can be reduced to O(*NlogN)* by MLFMA. The iterative solution of (1) usually converges slowly due to the ill-conditioned FEM matrix

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in (1). Hence, the key to design an efficient algorithm for (1) is how to construct a good preconditioner for the FEM part of (1).

To construct a preconditioner for the hierarchical higher-order FEM matrix of (1), the multiplicative Schwarz method is employed first in this paper. To be more specific, the FEM matrix part in (1) is rewritten by classifying unknowns into different groups corresponding to different FEM orders. Taking the hierarchical second-order FEM matrix as example, namely, the following FEM matrix in (1)

$$K = \begin{bmatrix} K_{II} & K_{IS} \\ K_{SI} & K_{SS} \end{bmatrix} \tag{2}$$

is rewritten as

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{3}$$

where subscript "1" corresponds to the first-order FEM unknowns, "2" to the second-order FEM unknowns. Then this rewritten FEM matrix is factorized as

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ K_{21}K_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} - K_{21}K_{11}^{-1}K_{12} \end{bmatrix} \begin{bmatrix} I & K_{11}^{-1}K_{12} \\ 0 & I \end{bmatrix} \tag{4}$$

Based on this factorization, the preconditioning matrix of [K] can be approximated, according to the multiplicative Schwarz method, as

$$M = \begin{bmatrix} I & 0 \\ K_{21}K_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix} \begin{bmatrix} I & K_{11}^{-1}K_{12} \\ 0 & I \end{bmatrix} \tag{5}$$

Since the inverse of [M] can be directly written out, a good preconditioner of [K] can be obtained as

$$M^{-1} = \begin{bmatrix} I & -K_{11}^{-1}K_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} K_{11}^{-1} & 0 \\ 0 & K_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -K_{21}K_{11}^{-1} & I \end{bmatrix} \tag{6}$$

The above construction procedure of the second-order hierarchical FEM matrix can be extended to any order FEM matrix. Since the principle of grouping is based on the order of unknowns, this preconditioner is called as p-type multiplicative Schwarz (p-MUS) preconditioner. It is worth to point out that (5) is obtained by dropping coupling term of $-K_{21}K_{11}^{-1}K_{12}$. Thus, weaker the coupling of the different order basis functions, more efficient the p-MUS preconditioner. In practice, the hierarchical basis functions are orthogonalized before constructing p-MUS preconditioner.

It is known from (6) that to obtain the preconditioner, namely M^{-1} , the inverses of K_{11} and K_{22} are required. Since completely computing inverses of K_{11} and K_{22} is still expensive, we usually compute approximations of the inverses of K_{11} and K_{22} instead. In this paper, we utilize an h-type multiplicative Schwarz (h-MUS) to obtain the approximations of the inverses of K_{11} and K_{22} instead of utilizing conventional incomplete LU factorization. Therefore, we name the so-obtained preconditioner as the h-p-MUS preconditioner.

III. Numerical Results

To demonstrate efficiency of the presented preconditioning algorithm described above, we compute scattering by a dielectric sphere. Table 1 lists the fill-in factors in the construction of the p-MUS preconditioner and h-p-MUS preconditioner and the CPU time required in these preconditioned solutions. It can be seen from table 1 that the fill-in factor in the h-p-MUS preconditioner construction is about half of that in the p-MUS preconditioner construction. The CPU time required in the h-p-MUS preconditioner construction is about 25% less than that in the p-MUS precondtioner construction. Furthermore, the h-p-MUS preconditioner performs better than the p-MUS preconditioner. Hence, the total CPU time in the h-p-MUS preconditioned solution is about 40% less than that of the p-MUS preconditioned solution.

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Table 1 Numerical performance of the h-p-MUS versus the p-MUS for the dielectric sphere

Order	Unknowns	Nonzeros	Fill-in factor		CPU time (s)		Number of iterations	
			p-MUS	h-p-MUS	p-MUS	h-p-MUS	p-MUS	h-p-MUS
1st	17024	269312	22.0	10.6	14.9	8.1	37	26
2nd	73472	2048000	7.0	4.0	22.6	16.2		
3rd	171392	7328768	2.0	1.1	24.9	23.1		
Total	261888	9646080	3.6	2.0	197.7	114.6		